Degree, In-Degree, Out-Degree
Weighted Degree, Weighted In-Degree, Weighted Out-Degree

The network of cattle holdings and movements were first analysed to investigate the structure of the network and to calculate the main parameters. For each node the following measures were calculated: degree, in-degree and out-degree.

The node ***degree*** is the number of relations (edges) of the nodes. However, in the case of the directed networks, we distinguish between in-degree (number of incoming neighbours) and out-degree (number of outgoing neighbours) of a vertex.

Degree sum formula (also sometimes called the **handshaking lemma**),

$$\sum_{v \in V} \deg(v) = 2|E|$$

for a graph with vertex set *V* and edge set *E*. Both results were proven by Leonhard Euler (1736) in his famous paper on the Seven Bridges of Königsberg that began the study of graph theory.

For a vertex, the number of head ends adjacent to a vertex is called the *in-degree* of the vertex and the number of tail ends adjacent to a vertex is its *out-degree* (called "branching factor" in trees).
Let $G = (V, E)$ and $v \in V$. The in-degree of v is denoted $\deg^{-}(v)$ and its out-degree is denoted $\deg^{+}(v)$. A vertex with $\deg^{-}(v) = 0$ is called a *source*, as it is the origin of each of its incident arrows. Similarly, a vertex with $\deg^{+}(v) = 0$ is called a *sink*. If a vertex is neither a *source* not a *sink*, it is called an *internal*.
The *degree sum formula* states that, for a directed graph,

$$\sum_{v \in V} \deg^{+}(v) = \sum_{v \in V} \deg^{-}(v) = |A|.$$

If for every vertex $v \in V$, $\deg^{+}(v) = \deg^{-}(v)$, the graph is called a *balanced directed graph.*

Degree has generally been extended to the sum of weights when analysing weighted networks and labelled node strength, so the ***weighted degree*** and the ***weighted in-*** and ***out-degree*** was calculated (Barrat et al. 2004, Newman 2001, Opsahl et al. 2010).

Barrat, A., Barthélémy, M., Pastor-Satorras, R. and Vespignani, A. 2004. The architecture of complex weighted networks. *Proceedings of the National Academy of Sciences* 101 (11): 3747–3752.

Newman, M. E. J. 2001. Scientific collaboration networks. II. Shortest paths, weighted networks, and centrality. *Physical Review E.* 64 (1): 1–7.

Opsahl, T., Agneessens, F. and Skvoretz, J. 2010. Node centrality in weighted networks: Generalizing degree and shortest paths. *Social Networks* 32 (3): 245–251.

Authority Score, Hub Score

The HITS algorithm was developed by Kleinberg (1999, 2000). This algorithm is a link analysis algorithm that helps in identifying the essential nodes in a graph. It consists of two scores, a ***hub score*** and an ***authority score***. The authority score of a node is a measure of the amount of valuable information that this node holds. The hub score of a node shows how many highly informative nodes or authoritative nodes this node is pointing to. So a node with a high hub score shows that this node is pointing to many other authoritative nodes. On the other hand, a node with a high authoritative score shows that it is pointed to a large number of nodes, thus serves as a node of useful information in the network.

The scheme therefore assigns two scores for each page: its authority, which estimates the value of the content of the page, and its hub value, which estimates the value of its links to other pages. Mathematically, these two centrality values are expressed as follows. The authority centrality of a node ( ) is proportional to the sum of hub centralities of nodes ( ) pointing to it, and is defined as

$$x_i = \alpha \sum_j A_{ji} y_j$$

The hub centrality of a node is proportional to the sum of authority centralities of nodes pointing to it, and is defined as

$$y_i = \beta \sum_j A_{ij} x_j$$

Kleinberg, J. M. 1999. Authoritative sources in a hyperlinked environment. *Journal of the ACM* 46 (5): 604–632.

Kleinberg, J. M. 2000. Navigation in a small world. *Nature* 406 (6798): 845–845.

Betweenness Centrality

Betweenness centrality is even more important statistical property of a network. This is applied in a lot of real-world problems, such as finding influential people in a social network, finding crucial hubs in a computer network, finding border crossing points which have a largest traffic or trade flow. The betweenness centrality of a node is an indicator of its centrality or importance in the network. It is described as the number of shortest paths from all the vertices to all the other vertices in the network that pass through the node in consideration (Brandes 2001).

The betweenness of a vertex v in a graph ***G: = (V,E)*** with ***V*** vertices is computed as follows:

1. For each pair of vertices (*s*,*t*), compute the shortest paths between them.
2. For each pair of vertices (*s*,*t*), determine the fraction of shortest paths that pass through the vertex in question (here, vertex *v*).
3. Sum this fraction over all pairs of vertices (*s*,*t*).

More compactly the betweenness can be represented as:

$$C_B(v) = \sum_{s \neq v \neq t \in V} \frac{\sigma_{st}(v)}{\sigma_{st}}$$

where σ_{st} is total number of shortest paths from node s to node t and $\sigma_{st}(v)$ is the number of those paths that pass through v. The betweenness may be normalised by dividing through the number of pairs of vertices not including *v*, which for directed graphs is $(n-1)(n-2)$ and for undirected graphs is $(n-1)(n-2)/2$. For example, in an undirected star graph, the center vertex (which is contained in every possible shortest path) would have a betweenness of $(n-1)(n-2)/2$ (1, if normalised) while the leaves (which are contained in no shortest paths) would have a betweenness of 0.

Brandes, U. 2001. A Faster Algorithm for Betweenness Centrality. *Journal of Mathematical Sociology* 25 (2): 163–177.

Cloeseness Centrality

Closeness centrality indicates how long it will take for information from a given node to reach other nodes in the network. The smaller the value, the more central role the node plays in the network.

In connected graphs there is a natural distance metric between all pairs of nodes, defined by the length of their shortest paths. The **farness** of a node *x* is defined as the sum of its distances from all other nodes, and its closeness was defined by Bavelas as the reciprocal of the farness, that is:

$$C(x) = \frac{1}{\sum_y d(y,x)}.$$

Thus, the more central a node is the lower its total distance from all other nodes. Note that taking distances *from* or *to* all other nodes is irrelevant in undirected graphs, whereas in directed graphs distances *to* a node are considered a more meaningful measure of centrality, as in general (e.g., in, the web) a node has little control over its incoming links (Alex 1950, Sabidussi 1966)

Alex, B. 1950. Communication patterns in task-oriented groups. *J. Acoust. Soc. Am.* 22 (6): 725–730.

Sabidussi, G. 1966. The centrality index of a graph. *Psychometrika.* 31: 581–603.